ON TORSIONAL VIBRATIONS OF HALF-SPACE

(O KRUTIL'NYKH KOLEBANIIAKH POLUPROSTRANSTVA)

PMM Vol.25, No.1, 1961, pp. 159-163

Ia.S. UFLIAND
(Leningrad)

(Received October 14, 1960)

This paper deals with steady-state vibrations of an elastic half-space, generated by the rotation of a rigid cylinder. The area of contact between the half-space and the cylinder is circular. The exact solution of this problem has been obtained by Sagoci [1] with the use of spheroidal wave functions.

Here, another approach to this problem is presented. It is based on the concept of dual integral equations which are equivalent to a regular Fredholm's equation, admitting an effective approximate solution.

1. Formulation of the problem and its reduction to Fredholm's integral equation. Let a rigid die, connected with the half-space $z > 0$ on the circle with the radius a, be acted upon by the torsional moment

$$M = M_0 \operatorname{Re} e^{i(\nu t+\alpha)} \qquad (\nu \text{ is frequency of vibration}) \qquad (1.1)$$

It is easy to verify that all the equations of theory of elasticity are satisfied if only the component of the displacement vector in the direction of the ϕ-line (r, ϕ, z are cylindrical coordinates) is assumed different from zero:

$$u_\varphi = \operatorname{Re}(ue^{i(\nu t+\alpha)}) \qquad (1.2)$$

with the function $u(r, z)$ satisfying the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0, \qquad k = \nu\sqrt{\frac{\rho}{G}} \qquad (1.3)$$

where ρ is the density and G is the shear modulus.

On the boundary of the half-space the following conditions are to be satisfied:

$$u|_{z=0} = \varepsilon r, \qquad r < a; \qquad \frac{\partial u}{\partial z}\bigg|_{z=0} = 0, \qquad r > a \tag{1.4}$$

where ϵ is the complex amplitude of the angle of rotation of the die which, in this problem, is considered to be known; later it will be expressed in terms of the given amplitude M_0 of the applied moment.

The second condition (1.4) means that the shear stress

$$\tau_{\varphi z} = G\frac{\partial u_\varphi}{\partial z} \tag{1.5}$$

is equal to zero on the surface outside of the die.

If the solution of Equation (1.3), which approaches zero for $z \to \infty$, is assumed in the form

$$u = \int_0^\infty e^{-z\sqrt{\lambda^2-k^2}} J_1(\lambda r) A(\lambda)\, d\lambda \tag{1.6}$$

then the following two integral equations for the function $A(\lambda)$ are obtained from the boundary conditions:

$$\int_0^\infty A(\lambda) J_1(\lambda r)\, d\lambda = \varepsilon r, \qquad r < a; \qquad \int_0^\infty \sqrt{\lambda^2 - k^2}\, J_1(\lambda r) A(\lambda)\, d\lambda = 0, \qquad r > a \tag{1.7}$$

Following the method of Lebedev [2] we obtain

$$A = \frac{\lambda}{\sqrt{\lambda^2 - k^2}} \int_0^a \varphi(t) \sin \lambda t\, dt \tag{1.8}$$

where $\phi(t)$ is a new unknown function. Using the relation [3]

$$\int_0^\infty J_0(\lambda r) \sin \lambda t\, d\lambda = 0, \qquad t < r \tag{1.9}$$

it can be shown that Equation (1.3) is then identically satisfied.

Substituting (1.8) into the first of Equations (1.7), and using the relation [3]

$$\int_0^\infty J_1(\lambda r) \sin \lambda t\, d\lambda = \frac{t}{r\sqrt{r^2 - t^2}}, \qquad t < r \tag{1.10}$$

the integral equation of Schlemilch can be obtained

$$\int_0^{\pi/2} F(r \sin \theta)\, d\theta = \varepsilon r^2 \tag{1.11}$$

where

$$F(x) = x\left\{\varphi(x) - \frac{1}{\pi}\int_0^a \varphi(t)\,[\psi(t-x) - \psi(t+x)]\,dt\right\} \tag{1.12}$$

and

$$\psi(y) = \int_0^\infty \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - k^2}}\right)\cos\lambda y\,d\lambda \tag{1.13}$$

The solution of Equation (1.11) has the form [4]

$$F(x) = \frac{4\varepsilon}{\pi}x^2 \tag{1.14}$$

The function $\psi(y)$ can be written explicitly [5]

$$\psi(y) = \frac{\pi k}{2}\left[J_1(k\,|\,y\,|) - iH_1(ky) + \frac{2i}{\pi}\right] \tag{1.15}$$

where $H_1(z)$ is the Struve function. Thus, the problem reduces to the solution of a regular Fredholm equation

$$\varphi(x) - \frac{1}{\pi}\int_0^a \varphi(t)\,[\psi(t-x) - \psi(t+x)]\,dt = \frac{4\varepsilon}{\pi}x, \quad 0 < x < a \tag{1.16}$$

whose kernel is given by Expression (1.15).

2. Numerical calculations. Assuming

$$\varphi(x) = \frac{4\varepsilon a}{\pi}\omega(\xi), \qquad \xi = \frac{x}{a}, \qquad \tau = \frac{t}{a}, \qquad \psi(y) = \frac{\pi k}{2a}L(y) \tag{2.1}$$

we reduce the fundamental equation (1.16) to the dimensionless form

$$\omega(\xi) = \xi + \frac{p}{2}\int_0^1 \omega(\tau)\,[L(\tau - \xi) - L(\tau + \xi)]\,d\tau \tag{2.2}$$

where

$$L(y) = J_1(p\,|\,y\,|) - iH_1(py) + \frac{2i}{\pi}, \quad p = ka \tag{2.3}$$

For the numerical solution of Equation (2.2) we assume

$$\omega(\xi) = \lambda(\xi) + i\mu(\xi) \tag{2.4}$$

and for the unknown functions $\lambda(\xi)$ and $\mu(\xi)$ we obtain the following system of real integral equations:

$$\lambda(\xi) = \xi + \frac{p}{2}\int_0^1 \lambda(\tau)\{J_1[ka\,|\tau-\xi\,|] - J_1[ka(\tau+\xi)]\}\,d\tau + \\ + \frac{p}{2}\int_0^1 \mu(\tau)\{H_1[ka(\tau-\xi)] - H_1[ka(\tau+\xi)]\}\,dt$$

$$\mu(\xi) = \frac{p}{2}\int_0^1 \lambda(\tau)\{H_1[ka(\tau+\xi)] - H_1[ka(\tau-\xi)]\}\,d\tau + \\ + \frac{p}{2}\int_0^1 \mu(\tau)\{J_1[ka\,|\tau-\xi\,|] - J_1[ka(\tau+\xi)]\}\,d\tau \qquad (2.5)$$

The system (2.5) has been solved numerically, replacing it by an algebraic system. The results of the calculation (which are obtained by dividing the interval into ten sections for various values of the parameter p) are given in Table 1.

TABLE 1.

τ	$p = 0.4$		$p = 0.8$		$p = 1.2$		$p = 1.6$		$p = 2$	
	λ	μ	λ	μ	λ	μ	λ	μ	λ	μ
0.1	0.0962	0.00084	0.0870	0.0055	0.0753	0.0144	0.0624	0.0266	0.0485	0.0418
0.2	0.1925	0.00168	0.1743	0.0110	0.1514	0.0286	0.1264	0.0530	0.0996	0.0830
0.3	0.2890	0.00256	0 2623	0.0164	0.2291	0.0428	0.1932	0.0789	0.1554	0.1231
0.4	0.3857	0.00335	0.3514	0.0220	0.3092	0.0568	0.2645	0.1042	0.2183	0.1616
0.5	0.4828	0.00419	0.4417	0.0273	0.3923	0.0705	0.3416	0.1286	0.2906	0.1981
0.6	0.5803	0.00502	0.5338	0.0325	0.4793	0.0839	0.4260	0.1520	0.3743	0.2321
0.7	0.6782	0.00585	0.6279	0.0379	0.5714	0.0969	0.5187	0.1742	0.4715	0.2632
0.8	0.7768	0.00668	0.7243	0.0430	0.6690	0.1095	0.6215	0.1951	0.5841	0.2911
0.9	0.8759	0.00751	0.8234	0.0481	0.7723	0.1217	0.7352	0.2144	0.7138	0.3156
1.0	0.9758	0.00832	0.9256	0.0532	0.8826	0.1332	0.8609	0.2322	0.8620	0.3364

For the complete solution of this problem it is necessary to find the relation between the given torsional moment M_0 and the complex amplitude ϵ of the angle of rotation of the die.

Using the equation

$$M = \int_0^a\int_0^{2\pi} \tau_{\varphi z}\,|_{z=0} r^2\,dr\,d\varphi = 2\pi G\,\mathrm{Re}\,e^{i(\nu t+\alpha)}\int_0^a \frac{\partial u}{\partial z}\bigg|_{z=0} r^2\,dr$$

and evaluating the integral

$$\int_0^a \frac{\partial u}{\partial z}\Big|_{z=0} r^2\,dr = \int_0^a r^2\,dr \int_0^\infty \lambda J_1(\lambda r)\,d\lambda \int_0^a \varphi(t)\sin\lambda t\,dt$$

with the use of Equation [3]

$$\int_0^\infty J_0(\lambda r)\sin\lambda t\,d\lambda = \frac{1}{\sqrt{t^2 - r^2}}, \qquad t > r \tag{2.6}$$

we obtain the following expression:

$$M = M_0 \operatorname{Re} e^{i(\nu t+\alpha)} = -16Ga^3 \operatorname{Re} \varepsilon \int_0^1 \tau\omega(\tau)\,d\tau \tag{2.7}$$

Hence, the necessary relation is obtained

$$\varepsilon = -M_0 \{16a^3 G \int_0^1 [\lambda(\tau) + i\mu(\tau)]\,\tau\,d\tau\}^{-1} \tag{2.8}$$

TABLE 2.

p	$\varkappa$	G_1	G_2
0.4	0.00861	0.6038	0.00520
0.8	0.05962	0.6462	0.03851
1.2	0.1651	0.6859	0.1132
1.6	0.3190	0.6922	0.2208
2.0	0.5127	0.6479	0.3922

To compare our results with those of Sagoci [1], we write the relation (2.8) in the form

$$\varepsilon = \frac{9M_0}{16\pi a^3 G}\frac{\gamma + i\beta}{\gamma^2 + \beta^2} \tag{2.9}$$

where

$$\beta = \frac{9}{\pi}\int_0^1 \mu(\tau)\,\tau\,d\tau, \qquad \gamma = -\frac{9}{\pi}\int_0^1 \lambda(\tau)\,\tau\,d\tau \tag{2.10}$$

Table 2 shows the values of the quantities

$$G_1 = -\frac{9}{16}\frac{\gamma}{\gamma^2+\beta^2}, \qquad G_2 = \frac{9}{16}\frac{\beta}{\beta^2+\gamma^2}, \qquad \varkappa = -\frac{\beta}{\gamma} \tag{2.11}$$

which agree with the corresponding numerical results of Sagoci.

The author expresses his gratitude to K.A. Aristova and T.A. Chernova who performed the numerical calculations.

BIBLIOGRAPHY

1. Sagoci, H.F., Forced torsional oscillations of an elastic half-space, II. *J. Appl. Phys.* Vol. 15, No. 9, 1944.

2. Lebedev, N.N., Raspredelenie elektrichestva na tonkom paraboloidal'nom segmente (Distribution of electricity on a thin paraboloidal segment). *Dokl. Akad. Nauk SSSR* Vol. 114, No. 3, 1957.

3. Watson, G.N., *Besselevy funktsii (Bessel Functions)*. IIL, 1948.

4. Whittaker, E.T. and Watson, G.N., *Kurs sovremennogo analiza (Modern Analysis)*. GTTI, 1933.

5. Lebedev, N.N. and Skal'skaia, I.P., Novyi metod resheniia zadachi difraktsii elektromagnitnykh voln na tonkom provodiashchem diske (New method of solution of the problem of diffraction of electromagnetic waves on a thin conducting disc). ZhTF Vol. 29, No. 6, 1959.

Translated by M.P.B.